Journal of Approximation Theory **99**, 409–421 (1999)
Article ID jath.1999.3327, available online at http://www.idealibrary.com on IDEAL®

Comonotone Jackson's Inequality

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Communicated by Dany Leviatan

Received February 25, 1997; accepted in revised form November 5, 1998

Let $2s$ points $y_i = -\pi \leqslant y_{2s} < \cdots < y_1 < \pi$ be given. Using these points, we define the points y_i for all integer indices i by the equality $y_i = y_{i+2s} + 2\pi$. We shall write $f \in \Delta^{(1)}(Y)$ if f is a 2π-periodic continuous function and f does not decrease on $[y_i, y_{i-1}]$, if i is odd; and f does not increase on $[y_i, y_{i-1}]$, if i is even. In this article the following Theorem 1—the comonotone analogue of Jackson's inequality—is proved.

THEOREM 1. *If $f \in \Delta^{(1)}(Y)$, then for each nonnegative integer n there is a trigonometric polynomial $\tau_n(x)$ of order $\leqslant n$ such that $\tau_n \in \Delta^{(1)}(Y)$, and $|f(x) - \pi_n(x)| \leqslant c(s)\, \omega(f; 1/(n+1))$, $x \in \mathbb{R}$, where $\omega(f; t)$ is the modulus of continuity of f, $c(s) = const$. Depending only on s.*

Key Words: comonotone approximation; polynomial approximation.

1. INTRODUCTION

Let $\mathbb{T}_n$ be the space of trigonometric polynomials

$$\tau_n(x) := a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$$

of order $\leqslant n$.

We denote by

$$\|f\| := \max_{x \in \mathbb{R}} |f(x)|,$$

$\omega(f; t)$-modulus of continuity of f.

A continuous 2π-periodic function f is called *bell-shaped*, if it is even and decreases on $[0, \pi]$.

In 1968 Lorentz and Zeller [11] proved the following Theorem LZ.

0021-9045/99 $30.00

THEOREM LZ. *There exists a constant c with the following property. For each bell-shaped function f, one can find a bell-shaped trigonometric polynomial τ_n for which*

$$\|f-\tau_n\| \leqslant c\omega, \left(f; \frac{1}{n}\right).$$

In the other words, Jackson's inequality remains true, if we approximate bell-shaped functions by bell-shaped polynomials.

In the same paper [11], they applied Theorem LZ to the estimate of approximation of monotone and continuous functions on $[-1, 1]$ by monotone algebraic polynomials on $[-1, 1]$. Since then, the problem of uniform and pointwise approximation of monotone and piecewise monotone functions f on a segment by comonotone algebraic polynomials has been deeply investigated by (alphabetically) Beatson, DeVore, Ditzian, Dzyubenko, Gilewicz, Hu, Iliev, Kopotun, Leviatan, Lorentz, Newman, Passow, Raymon, Roulier, Shevchuk, Shvedov, Wu, Yu, Zeller, Zhou, and others, see [1, 4–11, 13, 14] for the references. At the same time the author does not know any similar results for the periodic case, except Theorem LZ. Koniagin and Demidovich have attracted attention to the "periodic" case in their talk with Shevchuk, who has offered to the author to investigate this problem. A possible reason of absence of "periodic" results is the following. The problem of approximation of monotone continuous periodic functions is not interesting since these functions are identical constants, but the methods of piecewise-monotone approximation by algebraic polynomials have only recently advanced.

Let $2s$ points y_i

$$-\pi \leqslant y_{2s} < y_{2s-1} < \cdots < y_1 < \pi$$

be given. Starting from these points, we define by

$$y_i = y_{i+2s} + 2\pi$$

the points y_i for all integer indexes i, in particular, $y_0 = y_{2s} + 2\pi$, $y_{2s+1} = y_1 - 2\pi$, etc. We denote $Y := \{y_i\}_{i \in \mathbb{Z}}$. We shall write $f \in \Delta^{(1)}(Y)$ if f is a 2π-periodic continuous function which does not decrease on $[y_i, y_{i-1}]$, if i is odd, and does not increase on $[y_i, y_{i-1}]$, if i is even.

Remark. The definition of the set $\Delta^{(1)}(Y)$ used the fact that the number of initial points y_i is even. Similarly, it is possible to define formally the set $\Delta^{(1)}(Y)$ for odd number of initial points. It is easy to notice, however, that in this case the set $\Delta^{(1)}(Y)$ would consist of identical constants only, that is not interesting.

We denote

$$\Pi(t) := \prod_{i=1}^{2s} \sin \tfrac{1}{2}(t - y_i),$$

and observe that $\Pi \in \mathbb{T}_s$, that is, Π is a trigonometric polynomials of order s. Since $(t - y_i) \sin\frac{1}{2}(t - y_i) > 0$, $t \in [-\pi, \pi)$, $y_i \in [-\pi, \pi)$, $t \neq y_i$, for a 2π-periodic differentiable function f, the condition $f \in \Delta^{(1)}(Y)$ is equivalent to the inequality

$$f'(t)\, \Pi(t) \geqslant 0, \qquad t \in \mathbb{R}.$$

Now we are in a position to formulate the main result of the paper—Theorem 1, and the known result—Theorem LS. Theorem LS is a corollary of results for comonotone approximation by algebraic polynomials in the papers by Leviatan [8] and Leviatan and Shevchuk [10], it can be derived (using change of variables $x = \cos t$), say, as a corollary of Theorem 1 in [10].

THEOREM LS. *Let the set Y be symmetric about the origin, and $0 \in Y$ and $\pi \in Y$. If $f \in \Delta^{(1)}(Y)$ is even, then for each nonnegative integer n there is a trigonometric polynomial τ_n of order $\leqslant n$ such that*

$$\tau_n \in \Delta^{(1)}(Y)$$

and

$$\|f - \tau_n\| \leqslant c(s)\, \omega\left(f; \frac{1}{n+1}\right),$$

where $c(s) = const$, depends only on s.

Modifying for the periodic case the arguments of comonotone approximation by algebraic polynomials, we prove Theorem 1, which holds for arbitrary Y and $f \in \Delta^{(1)}(Y)$.

THEOREM 1. *If $f \in \Delta^{(1)}(Y)$, then for each nonnegative integer n there is a trigonometric polynomial τ_n of order $\leqslant n$ such that*

$$\tau_n \in \Delta^{(1)}(Y)$$

(*that is,*

$$\tau_n'(x)\, \Pi(x) \geqslant 0, \qquad x \in \mathbb{R})$$

and

$$\|f-\tau_n\| \leqslant c(s)\, \omega\left(f; \frac{1}{n+1}\right),$$

where $c(s) = const$ *depends only on* s.

Remark. In Theorem LS it is possible to replace

$$\|f-\tau_n\| \leqslant c(s)\, \omega\left(f; \frac{1}{n+1}\right) \tag{1}$$

by

$$\|f-\tau_n\| \leqslant c(Y)\, \omega_2\left(f; \frac{1}{n+1}\right),$$

where $c(Y) = const$, depends only on Y, and $\omega_k(f; t)$ is the modulus of smoothness of order k. One can deduce this as a corollary of the result for comonotone approximation by algebraic polynomials in the paper by Kopotun and Leviatan [7] and the result of Djuzhenkova [2]. We do not know now whether the same substitution is possible in Theorem 1. At the same time we have a counterexample [12] that shows that in Theorem 1 it is impossible to replace (1) with

$$\|f-\tau_n\| \leqslant c(Y)\, \omega_k\left(f; \frac{1}{n+1}\right),$$

for $k > 2$. In our forthcoming paper the analog of the second Jackson's inequality, that is, for differentiable functions, shall be proved. Proof of Theorem 2 is too long to be included in this paper. The assumption about differentiability of function f makes it possible to formulate this analog in a simpler form than Theorem 1.

THEOREM 2. *If* 2π*-periodic* r *times continuously differentiable function* f *has on the period a finite number of changes of monotonicity, then for each positive integer* n *there is a trigonometric polynomial* τ_n *of order* $\leqslant n$ *such that*

$$f'(x)\, \tau_n'(x) \geqslant 0, \qquad x \in \mathbb{R}, \tag{2}$$

and

$$\|f-\tau_n\| \leqslant c(s)\, \frac{1}{(n+1)^r}\, \omega\left(f^{(r)}; \frac{1}{n+1}\right), \tag{3}$$

where $c(s) = const$ *does not depend on* f *and* n.

2. PROOF OF THEOREM 1

Theorem 1 follows from the trivial inequality $\|f-f(0)\| \leqslant \omega(f;\pi)$ and the following Theorem 3.

Everywhere below we denote by $c_1, c_2, \ldots$ constants, which depend only on s.

THEOREM 3. *If $f \in \Delta^{(1)}(Y)$, then for each integer $n > c_1$ there is a trigonometric polynomial $\tau_{(s+2)n}$ of order $<(s+2)\,n$ such that*

$$\tau_{(s+2)n} \in \Delta^{(1)}(Y) \tag{4}$$

and

$$\|f-\tau_{(s+2)n}\| \leqslant c_2 \omega\left(f; \frac{1}{n}\right). \tag{5}$$

The proof of Theorem 3 we divide into five items.

(1°) We denote by

$$J_n(x) = \frac{1}{\gamma_n}\left(\frac{\sin(nx/2)}{\sin(x/2)}\right)^{2(s+2)}$$

the Jackson type kernel (see S. B. Stechkin [15]), where

$$\gamma_n = \int_{-\pi}^{\pi}\left(\frac{\sin(nt/2)}{\sin(t/2)}\right)^{2(s+2)} dt.$$

The following inequalities are well known (see, for example, V. K. Dzjadyk [3, pp. 130, 131], I. A. Shevchuk [13, p. 128])

$$\int_{-\pi}^{\pi} (1+n\,|t|)^{2s}\, J_n(t)\, dt \leqslant c_3, \tag{6}$$

where $c_3 > 1$,

$$\frac{1}{c_4} n^{2s+3} \leqslant \gamma_n \leqslant c_4 n^{2s+3}.$$

Evidently,

$$J_n \in \mathbb{T}_{(s+2)(n-1)}.$$

(2°) We fix an integer

$$n > 4s^2 c_3 + s =: c_1, \tag{7}$$

and for all $j \in \mathbb{Z}$, set

$$x_j := -\frac{\pi j}{n};$$

$$\delta_j(x) := \min\left\{1; \frac{1}{n\,|\sin((x - x_j)/2)|}\right\}.$$

For $x \in [x_j, x_j + 2\pi]$ we have, with $t = \min\{x - x_j; x_j + 2\pi - x\}$,

$$n\left|\int_x^{x_j+\pi} \delta_j^4(u)\,du\right| = n\int_t^{\pi} \min\left\{1; \frac{1}{n^4 \sin^4(u/2)}\right\} du$$

$$\leqslant c_5 \min\left\{1; \frac{1}{n^3\,|\sin^3(t/2)|}\right\} = c_5 \delta_j^3(x),$$

which implies, for $x \in [x_j - 2\pi, x_j]$,

$$n\left|\int_x^{x_j-\pi} \delta_j^4(u)\,du\right| = n\left|\int_{x+2\pi}^{x_j+\pi} \delta_j^4(u)\,du\right|$$

$$\leqslant c_5 \delta_j^3(x + 2\pi) = c_5 \delta_j^3(x). \tag{8}$$

We shall use the estimate

$$\left\|\sum_{j=1-n}^{n} \delta_j^2\right\| < 6. \tag{9}$$

To prove (9) we fix j_*, $x \in [x_{j_*+1}, x_{j_*}]$ and get

$$\sum_{j=1-n}^{n} \delta_j^2(x) = \sum_{j=j_*+1-n}^{j_*+n} \delta_j^2(x) \leqslant 2 + \sum_{j=j_*+2}^{j_*+n} \delta_j^2(x) + \sum_{j=j_*+1-n}^{j_*-1} \delta_j^2(x)$$

$$\leqslant 2 + \left(\frac{\pi}{n}\right)^2 \sum_{j=j_*+2}^{j_*+n} \frac{1}{(x - x_j)^2} + \left(\frac{\pi}{n}\right)^2 \sum_{j=j_*+1-n}^{j_*-1} \frac{1}{(x - x_j)^2}$$

$$\leqslant 2 + 2\left(\frac{\pi}{n}\right)^2 \left(\frac{n}{\pi}\right)^2 \sum_{j=1}^{\infty} \frac{1}{j^2} < 6.$$

We shall write $j \in H_*$, if

$$\min_{i \in \mathbb{Z}} |x_j - y_i| \geqslant 2sc_3 \frac{\pi}{n} =: c_6 \frac{\pi}{n};$$

and $j \in H$, if $j \in H_*$, $|j| \leqslant n$, and $j \neq -n$.

Remark. Observe the number of indices $j \bar{\in} H_*$ such that $|j| \leqslant n$ and $j \neq -n$ does not exceed $2c_1$. Therefore assumption (7) means that H_* and H are not empty.

We shall use the simple estimates

$$\left|\sin \frac{x_j - y_i}{2}\right| \geqslant \frac{c_6}{n} = \frac{2sc_3}{n} > \frac{2s}{n} > \frac{1}{n}, \qquad j \in H_*, \quad i \in \mathbb{Z}; \tag{10}$$

$$\begin{aligned}\left|\frac{\sin((x - y_i)/2)}{\sin((x_j - y_i)/2)}\right| &= \left|\cos \frac{x - x_j}{2} + \frac{\sin((x - x_j)/2)}{\sin((x_j - y_i)/2)} \cos \frac{x_j - y_i}{2}\right| \\ &\leqslant 1 + n \left|\sin \frac{x - x_j}{2}\right| \\ &\leqslant \frac{2}{\delta_j(x)}, \qquad x \in \mathbb{R}, \quad j \in H_*, \quad i \in \mathbb{Z}.\end{aligned} \tag{11}$$

For each $j \in H_*$ denote by

$$\begin{aligned}\tilde{T}_j(x) &:= \int_{x_j - \pi}^{x} J_n(t - x_j) \frac{\Pi(t)}{\Pi(x_j)} \, dt, \\ d_j &:= \tilde{T}_j(x_j + \pi); \\ T_j(x) &:= \frac{1}{d_j} \tilde{T}_j(x).\end{aligned}$$

(3°) Let us prove two lemmas.

LEMMA 1. *If $j \in H_*$, then*

$$\tfrac{1}{2} < d_j < \tfrac{3}{2}. \tag{12}$$

Proof. For each x we have

$$1 - \frac{\Pi(x_j + x)}{\Pi(x_j)} = -\frac{x}{2} \sum_{i=1}^{2s} \frac{\cos((\theta - y_i)/2)}{\sin((x_j - y_i)/2)} \prod_{\nu = 1, \nu \neq i}^{2s} \frac{\sin((\theta - y_\nu)/2)}{\sin((x_j - y_\nu)/2)}$$

with some θ, lying between $x + x_j$ and x_j. Assumption $j \in H_*$ and the estimate (10) imply

$$\left|\frac{\cos((\theta - y_i)/2)}{\sin((x_j - y_i)/2)}\right| \leqslant \frac{n}{c_6} = \frac{n}{2sc_3},$$

whereas (11) yields

$$\left|\frac{\sin((\theta-y_\nu)/2)}{\sin((x_j-y_\nu)/2)}\right| \leqslant 1+n\left|\sin\frac{\theta-x_j}{2}\right| \leqslant 1+n\,|x|.$$

Hence

$$\left|1-\frac{\Pi\,(x_j+x)}{\Pi\,(x_j)}\right| \leqslant \frac{|x|}{2}\,\frac{n}{2sc_3}\,2s(1+n\,|x|)^{2s-1}$$

$$< \frac{1}{2c_3}\,(1+n\,|x|)^{2s}, \qquad x\in\mathbb{R}.$$

Therefore by (6)

$$|1-d_j| = \left|\int_{-\pi}^{\pi} J_n(t)\left(1-\frac{\Pi\,(x_j+t)}{\Pi\,(x_j)}\right)dt\right|$$

$$< \frac{1}{2c_3}\int_{-\pi}^{\pi}(1+n\,|t|)^{2s}\,J_n(t)\,dt \leqslant 1/2. \quad \blacksquare$$

For each $j\in\mathbb{Z}$ we denote

$$\chi_j(x) := \begin{cases} 0, & \text{if} \quad x\leqslant x_j, \\ 1, & \text{if} \quad x>x_j. \end{cases}$$

LEMMA 2. *If $j\in H$, then for each $x\in[-\pi,\pi]$*

$$|\chi_j(x)-T_j(x)| \leqslant c_7\delta_j^2(x). \tag{13}$$

Proof. Evidently,

$$J_n(x-x_j) \leqslant c_4 n\delta_j^{2(s+2)}(x).$$

Hence, (11) and (12) imply

$$|T_j'(x)| = \frac{1}{d_j}\,J_n(x-x_j)\left|\frac{\Pi\,(x)}{\Pi\,(x_j)}\right| \leqslant 2^{2s+1}c_4 n\delta_j^4(x), \qquad x\in\mathbb{R}.$$

Therefore, if $x\in[-\pi,x_j]$, then

$$|\chi_j(x)-T_j(x)| = |T_j(x)| = \left|\int_{x_j-\pi}^{x} T_j'(t)\,dt\right|$$

$$\leqslant 2^{2s+1}c_4 n\left|\int_{x_j-\pi}^{x}\delta_j^4(t)\,dt\right| \leqslant c_7\delta_j^3(x),$$

where we used (8) in the last inequality; and if $x \in (x_j, \pi]$, then

$$|\chi_j(x) - T_j(x)| = |1 - T_j(x)| = \left| \int_x^{x_j+\pi} T_j'(t)\, dt \right|$$

$$\leqslant 2^{2s+1} c_4 n \left| \int_x^{x_j+\pi} \delta_j^4(t)\, dt \right| \leqslant c_7 \delta_j^3(x). \quad \blacksquare$$

Corollary. *For all $x \in [-\pi, \pi]$ the estimate*

$$\sum_{j \in H} |\chi_j(x) - T_j(x)| \leqslant c_8. \tag{14}$$

holds.

Indeed, the estimates (13) and (9) lead to

$$\sum_{j \in H} |\chi_j(x) - T_j(x)| \leqslant c_7 \sum_{j \in H} \delta_j^2(x) \leqslant c_7 \sum_{j=1-n}^{n} \delta_j^2(x) \leqslant c_8.$$

Remark. If $j \in H_*$, then

$$\Pi(x_j)\, T_j'(x)\, \Pi(x) \geqslant 0, \qquad x \in \mathbb{R}, \tag{15}$$

and since $T_j'(x)$ is a trigonometric polynomial with the constant term equal to

$$\frac{1}{2\pi} \int_{x_j-\pi}^{x_j+\pi} T_j'(t)\, dt = \frac{1}{2\pi},$$

then

$$T_j(x) = \frac{1}{2\pi} x + p_j(x), \tag{16}$$

where $p_j \in \mathbb{T}_{(s+2)(n-1)+s} \subset \mathbb{T}_{(s+2)n}$.

(4°) We put

$$V(x) := f(-\pi) + \sum_{j \in H} (f(x_{j-1}) - f(x_j))\, T_j(x). \tag{17}$$

Lemma 3. *If $f \in \Delta^{(1)}(Y)$, then*

$$V'(x)\, \Pi(x) \geqslant 0, \qquad x \in \mathbb{R}, \tag{18}$$

and for all $x \in [-\pi, \pi]$, *the estimate*

$$|f(x) - V(x)| \leqslant c_9 \omega\left(f; \frac{\pi}{n}\right) \tag{19}$$

holds.

Proof. The hypothesis $f \in \Delta^{(1)}(Y)$ yields

$$(f(x_{j-1}) - f(x_j))\, \Pi\,(x_j) \geqslant 0, \qquad j \in H,$$

that in view of (15), implies (18).

Now we prove (19). By (14) we have

$$\left|\sum_{j \in H} (f(x_{j-1}) - f(x_j))(T_j(x) - \chi_j(x))\right| \leqslant c_8 \omega\left(f; \frac{\pi}{n}\right), \quad x \in (-\pi; \pi]. \tag{20}$$

We let

$$S(x) := f(-\pi) + \sum_{j=1-n}^{n} (f(x_{j-1}) - f(x_j))\, \chi_j(x).$$

It is easy to verify the estimate

$$|S(x) - f(x)| \leqslant \omega\left(f; \frac{\pi}{n}\right), \qquad x \in [-\pi, \pi]. \tag{21}$$

Now we represent the difference $f(x) - V(x)$ in the form

$$f(x) - V(x) = f(x) - S(x) + \sum_{j \in H} (f(x_{j-1}) - f(x_j))(\chi_j(x) - T_j(x))$$
$$+ \sum_{j=1-n,\, j \notin H}^{n} (f(x_{j-1}) - f(x_j))\, \chi_j(x).$$

Since the last sum contains no more then $2c_1$ terms, it does not exceed

$$2c_1 \omega\left(f; \frac{\pi}{n}\right). \tag{22}$$

We combine (20), (21), and (22) to get (19) with the constant $c_9 = c_8 + 1 + 2c_1$. ∎

LEMMA 4. *The function* V *has the form*

$$V(x) = \frac{A}{2\pi} x + q(x), \tag{23}$$

where $q \in \mathbb{T}_{(s+2)(n-1)+s}$, *A is a constant, and*

$$|A| \leqslant 2c_1 \omega\left(f; \frac{\pi}{n}\right). \tag{24}$$

Proof. By (17) and (16) the equality (23) is valid with

$$q(x) = f(-\pi) + \sum_{j \in H} (f(x_{j-1}) - f(x_j))\, p_j(x)$$

and

$$A = \sum_{j \in H} (f(x_{j-1}) - f(x_j)).$$

Now we prove (24). To estimate the constant A we represent it in the form

$$A = \sum_{j=1-n}^{n} (f(x_{j-1}) - f(x_j)) - \sum_{j=1-n,\, j \bar{\in} H}^{n} (f(x_{j-1}) - (f(x_j)).$$

The first sum here is equal to $f(x_{-n}) - f(x_n) = f(\pi) - f(-\pi) = 0$, where we took into account the periodicity of the function f. The second sum contains no more than $2c_1$ terms, therefore (24) holds. ∎

(5°) Now we conclude the proof of Theorem 3. To this end we consider three cases.

(a) Suppose there are at least two numbers $J_+ \in H$ and $j_- \in H$ such that $\Pi(x_{j_+}) > 0$ and $\Pi(x_{j_-}) < 0$. We take $\tau_{(s+2)n}(x)$ in the following form

$$\tau_{(s+2)n}(x) = \begin{cases} V(x) - AT_{j_-}(x), & \text{if } A > 0, \\ V(x) - AT_{j_+}(x), & \text{if } A \leqslant 0. \end{cases}$$

Then (18) and (15) yield

$$\tau'_{(s+2)n}(x)\, \Pi(x) \geqslant 0$$

and since by (23) and (16)

$$\tau_{(s+2)n}(x) = \begin{cases} q(x) - Ap_{j_-}(x), & \text{if } A > 0, \\ q(x) - Ap_{j_+}(x), & \text{if } A \leqslant 0, \end{cases}$$

so $\tau_{(s+2)n} \in \mathbb{T}_{(s+2)n}$. Finally we use the inequalities (19), (24), and (13) and, say in the case $A > 0$, we get

$$\|f - \tau_{(s+2)n}\| = \|f - V + AT_{j_-}\| \leqslant (c_9 + 2c_1(c_7 + 1))\, \omega\left(f; \frac{\pi}{n}\right).$$

(b) Let $\Pi(x_j) > 0$ for every $j \in H$. Then for each $i \in \mathbb{Z}$ the estimate

$$y_{2i-1} - y_{2i} \leqslant (2c_6 + 1)\frac{\pi}{n} \tag{25}$$

holds. We fix $k \in \mathbb{Z}$ such that $f(y_{2k}) = \max_{x \in \mathbb{R}} f(x)$ and choose l satisfying $k - s < l \leqslant k$ and $f(y_{2l-1}) = \min_{x \in \mathbb{R}} f(x)$. We take

$$\tau_{(s+2)n}(x) := f(y_{2k}).$$

Then (25) yields

$$\begin{aligned} \|\tau_{(s+2)n} - f\| = \|f(y_{2k}) - f\| &\leqslant f(y_{2k}) - f(y_{2l-1}) \\ &\leqslant (f(y_{2k}) - f(y_{2k-1})) + \cdots + (f(y_{2l}) - f(y_{2l-1})) \\ &\leqslant s\omega\left(f; \frac{(2c_6+1)\pi}{n}\right) \leqslant (2c_6+2)\, s\omega\left(f; \frac{\pi}{n}\right). \end{aligned}$$

(c) If $\Pi(x_j) < 0$ for every $j \in H$, then we reason similar to (b).

Theorem 3 is proved.

ACKNOWLEDGMENT

The author expresses thanks to Professor I. A. Shevchuk for inspiring conversations and helpful suggestions.

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